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AN ELEMENTARY EXPOSITION OF FROBENIUS'S THEORY OF GROUP-CHARACTERS AND GROUP-DETERMINANTS

BY LEONARD EUGENE DICKSON

INTRODUCTION

IN a series of recent memoirs* in the *Berliner Sitzungsberichte*, Frobenius has developed an elaborate theory which has already attracted considerable attention, both on account of the beauty of the results and of its importance in various applications. For example, it furnishes a determination of the minimum number of variables upon which a given group can be represented as a linear group (Klein's normal problem). After the success which had attended the introduction of characters of a commutative group, particularly in the determination of all its subgroups,† it was a natural step for Frobenius to introduce characters of any finite group.

Burnside‡ has re-established the main results due to Frobenius and has obtained new results on certain continuous groups related to the theory (see §3 below). A simplification in the method and a generalization of the results have been made by the writer.§ The papers just cited naturally rely upon technical group theory.

The present paper presents the chief results due to Frobenius and follows his methods as closely as an elementary treatment will admit. Symbolic notations are avoided and the material is presented with considerable detail, a number of illustrative examples being worked out. The paper is practically self-contained, involving no dependence upon technical branches of mathematics. Of group-theory itself, only the most elementary knowledge is presupposed. This may be reduced to the mere definition of a group by omitting §6 and §12 (the example being solved otherwise in §15), without breaking the continuity of the paper; but the reader acquainted with the rudiments of group-theory will find in the concrete nature of those sections excellent discipline for the later general theory, as well as an important theorem not elsewhere proved. The paper should prove of special interest to students of the theory of determinants.

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* Über Gruppencharaktere, 1896, pp. 985–1021; Über die Primfactoren der Gruppendeterminante, 1896, pp. 1343–1382; Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, 1897, pp. 994–1015; 1899, p. 482; Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, 1898, pp. 501–515; Über die Composition der Charaktere einer Gruppe, 1899, pp. 330–339.

† Compare Weber's excellent treatment, *Algebra*, II, Zweiter Abschnitt, where references are given.

‡ *Proc. Lond. Math. Soc.*, vol. 29 (1898), pp. 207–224, 546–565.

§ *Trans. Amer. Math. Soc.*, vol. 3 (1902), pp. 285–301; *Bull. Amer. Math. Soc.*, vol. 9 (1902), pp. 394–401.

In the second edition of his *Algebra* (pp. 193–218), Weber gives an account of the simplest results of the theory, following more closely the first method of Frobenius. Weber defines a character as a system of numbers satisfying a rather complicated system of equations, (51) below. The present paper, following more closely the second method of Frobenius, introduces the characters from the standpoint of their application to group-determinants, the definitions being explicit and very simple.

PROPERTIES OF THE GROUP-MATRIX

1. To the h elements E (the identity), A, B, C, . . . of a finite group H, we make correspond h independent variables x_E, x_A, x_B, x_C, . . , in such a way that $x_L = x_{MN}$ if $L = MN$ in the group. By the *group-matrix* is meant

$$\begin{pmatrix} x_{EE^{-1}} & x_{EA^{-1}} & x_{EB^{-1}} & \cdots\cdots & x_{EQ^{-1}} & \cdots\cdots \\ x_{AE^{-1}} & x_{AA^{-1}} & x_{AB^{-1}} & \cdots\cdots & x_{AQ^{-1}} & \cdots\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{PE^{-1}} & x_{PA^{-1}} & x_{PB^{-1}} & \cdots\cdots & x_{PQ^{-1}} & \cdots\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \equiv (x_{PQ^{-1}}) \equiv (x). \tag{1}$$

The table of subscripts is the body of a left-hand multiplication table for the group H. Thus, for a cyclic group of order 3 composed of the elements E, A, B ($A^3 = E$, $B^2 = A$), the multiplication table and the corresponding group-matrix are, respectively,

$$\begin{array}{c|ccc} & E^{-1} = E & A^{-1} = B & B^{-1} = A \\ \hline E & EE^{-1} = E & EA^{-1} = B & EB^{-1} = A \\ A & AE^{-1} = A & AA^{-1} = E & AB^{-1} = B \\ B & BE^{-1} = B & BA^{-1} = A & BB^{-1} = E \end{array} \qquad \begin{pmatrix} x_E & x_B & x_A \\ x_A & x_E & x_B \\ x_B & x_A & x_E \end{pmatrix}.$$

The determinant $\Theta(x) \equiv |x_{PQ^{-1}}|$ of matrix (1) is called the *group-deter-*

minant of H. Let $\Theta_{P,Q}$ be the adjoint (first minor with proper sign) of the element $x_{PQ^{-1}}$ in the group-determinant; the minor is obtained by erasing the row defined by P and the column defined by Q^{-1}.

THEOREM. *The value of* $\Theta_{P,Q}$ *depends only upon the product* $PQ^{-1} \equiv C$ *and not upon* P *and* Q *separately.*

After the replacement of the products giving the subscripts in (1) by the equivalent single elements, the variable x_C occurs once and but once in each row (or column) of the matrix; for the h positions of x_C, the adjoint will be shown to have the same value. Consider first the adjoint $\Theta_{P,P}$ of the variable $x_{PP^{-1}}$ in the main diagonal:

$$\Theta_{P,P} \equiv |x_{RS^{-1}}| \qquad (R, S = E, A, B, \ldots, \text{ with } P \text{ excluded}).$$

Set $R = UP$, $S = VP$, so that $RS^{-1} = UPP^{-1}V^{-1} = UV^{-1}$. Then

$$\Theta_{P,P} = |x_{UV^{-1}}| \equiv \Theta_{E,E} \qquad (U, V = A, B, \ldots, \text{ with } E \text{ excluded}).$$

To show next that $\Theta_{P,Q} = \Theta_{C,E}$ if $PQ^{-1} = C$, we set

$$x_{CRS^{-1}} = x'_{RS^{-1}} \qquad (R, S = E, A, B, \ldots).$$

Since CR and R run simultaneously through the series $E, A, B, \ldots$ of the h elements of the group H,

$$\pm |x_{RS^{-1}}| = |x_{CRS^{-1}}| \equiv |x'_{RS^{-1}}| \qquad (R, S = E, A, B, \ldots).$$

To obtain the term $x_{PQ^{-1}} \equiv x_C$ in the determinant $|x_{CRS^{-1}}|$, we must take $R = S$, so that the corresponding term in the determinant $|x'_{RS^{-1}}|$ is in the main diagonal. But the adjoints of the latter terms have been shown to be equal. Hence the same is true for the former terms, so that $\Theta_{P,Q} = \Theta_{C,E}$.

In view of the preceding theorem, we may introduce the notation $\Theta_{PQ^{-1}}$ for $\Theta_{P,Q}$. The result may therefore be stated in the form

$$\text{(2)} \qquad \Theta_{PQ^{-1}} = \frac{1}{h} \frac{\partial \Theta(x)}{\partial x_{PQ^{-1}}}.$$

DEFINITION. The matrix $(\Theta_{PQ^{-1}})$ is the *adjoint matrix* of the group-matrix $(x_{PQ^{-1}})$.

EXAMPLE. For the above cyclic group $\{E, A, B\}$, the adjoint matrix is

$$\begin{pmatrix} \Theta_E & \Theta_B & \Theta_A \\ \Theta_A & \Theta_E & \Theta_B \\ \Theta_B & \Theta_A & \Theta_E \end{pmatrix} \equiv \begin{pmatrix} x_E^2 - x_A x_B & x_B^2 - x_E x_A & x_A^2 - x_E x_B \\ x_A^2 - x_E x_B & x_E^2 - x_A x_B & x_B^2 - x_E x_A \\ x_B^2 - x_E x_A & x_A^2 - x_E x_B & x_E^2 - x_A x_B \end{pmatrix}.$$

Formula (2) is now evident since $\Theta(x) \equiv x_E^3 + x_A^3 + x_B^3 - 3x_E x_A x_B$.

DEFINITION. By the *symmetry* of the group-matrix of a given group H is meant the property that from the order of the variables in its first row follows definitely the order of the variables in each of its rows.

The above theorem may now be stated in the following form:

The group-matrix and its adjoint matrix possess the same symmetry.

Thus, for the cyclic group of order 3, the elements of both the group-matrix and the adjoint matrix have the subscripts E, B, A in the first row; A, E, B in the second row; B, A, E in the third row.

2. Let $y_E, y_A, y_B, \ldots$ be a second set of h independent variables and consider the matrix $(y) \equiv (y_{PQ^{-1}})$ with the same table of subscripts as matrix (1). From the two systems of variables x_R, y we build a third system z_R as follows:

$$z_T = \sum_R x_R y_S \equiv \sum_S x_R y_S \qquad (RS = T). \tag{3}$$

In the first sum, R runs through the series $E, A, B, \ldots$ of the h elements of H, and for each R the element $R^{-1}T$ is to be taken for S. Similarly for the second sum. By the composition of the two matrices, we find that

$$(x)(y) = (z) \equiv (z_{P,Q}), \tag{4}$$

$$z_{P,Q} \equiv \sum_R x_{PR^{-1}} y_{RQ^{-1}} \qquad (R = E, A, B, \ldots).$$

Set $RQ^{-1} = S$. Then R and S run simultaneously through the series $E, A, B, \ldots$ of the h elements of H. Applying also the definition (3), we get

$$z_{P,Q} = \sum_S x_{PQ^{-1}S^{-1}} y_S = z_{PQ^{-1}} \qquad (S = E, A, B, \ldots).$$

Hence the matrix (z) *possesses the same symmetry as the matrices* (x), (y).

3. As to the content of the preceding theorems a suggestive illustration is afforded by the following property, not mentioned by Frobenius and not essential in his theory, but forming the foundation of the investigations of Burnside and the writer. Consider the linear homogeneous transformation X whose coefficients are the elements $x_E, x_A, \ldots$ of the matrix (x). By allowing the variables $x_E, x_A, \ldots$ to take in turn every set of values in a given field (domain of rationality) F such that the determinant $\Theta(x)$ is not zero, we obtain a set of transformations which form a *group*. Indeed, by the preceding theorems, the inverse of X belongs to the set and the compound YX

of the transformations Y, X having the respective matrices (y), (x) is a transformation Z having the matrix (z) and consequently belonging to the set.

In view of the fact that the matrices (1) form a group, whereas other matrices of similar type do not, the group-matrix (1) is evidently of special importance in the study of the group H.

THE IRREDUCIBLE FACTORS OF THE GROUP-DETERMINANT

4. Applying to the determinants of the matrices (4) the multiplication theorem, we get

$$(5) \qquad \Theta(z) = \Theta(x)\,\Theta(y), \qquad \text{if } (z) = (x)(y).$$

Let the homogeneous integral rational function $\Theta(x)$ be decomposed into irreducible factors, each a homogeneous integral rational function, possibly with irrational coefficients:

$$(6) \qquad \Theta(x) = \Phi^{e}\Phi'^{e'}\Phi''^{e''}\cdots\cdot, \qquad \Phi \equiv \Phi(x_E, x_A, x_B, \cdots\cdot),$$

and let f, f', f'', ... be the degrees of Φ, Φ', Φ'', ..., respectively. The diagonal terms and no others in $\Theta(x)$ are x_E. Hence $\Theta(x)$ reduces to x_E^h if zero be substituted for each variable except x_E; whence $\Phi(x)$ reduces to cx_E^f, where $c \neq 0$. By choice of the undetermined constant factor of $\Phi(x)$, we can make $c = 1$. Among the factors Φ, Φ', ..., occurs the linear factor $x_E + x_A + x_B + \cdots\cdot$; indeed, this sum occurs throughout the first row of a determinant derived from $\Theta(x)$ upon adding to the first row of the latter all the remaining rows.

5. THEOREM. *The irreducible factors of the group-determinant* $\Theta(x)$ *possess the characteristic property*

$$(7) \qquad \Phi(z) = \Phi(x)\Phi(y), \qquad \text{if } (z) = (x)(y).$$

If $\Phi(z)$ is an irreducible factor of $\Theta(z)$, it must, in view of (5), equal the product of a function $\Phi_1(x)$ of x_E, x_A, ... alone and a function $\Phi_2(y)$ of y_E, y_A, ... alone. Taking $x_E = 1$, $x_A = x_B = \cdots = 0$, we get $(z) = (y)$, whence $\Phi(y) = \Phi_1(1, 0, 0, \cdots)\Phi_2(y)$. Taking next $y_E = 1$, $y_A = y_B = \cdots = 0$, we get $(z) = (x)$, whence $\Phi(x) = \Phi_1(x)\Phi_2(1, 0, 0, \ldots)$. By choice of the undetermined constant factor of $\Phi_1(x)$, we can make $\Phi_1(1, 0, 0, \cdots) = 1$. Since $c \equiv \Phi(1, 0, 0, \cdots) = 1$, we have $\Phi_2(1, 0, 0, \cdots) = 1$. Hence $\Phi_1(x) = \Phi(x)$, $\Phi_2(y) = \Phi(y)$, so that (7) follows.

Inversely, let $\Phi(x_E, x_A, \ldots)$ be an irreducible homogeneous integral rational function which satisfies relation (7). Taking for (y) the adjoint matrix to (x), we get $z_E = \Theta(x)$, $z_A = z_B = \cdots = 0$, whence $\Phi(z) = [\Theta(x)]^f$, and $\Phi(y)$ becomes an integral rational function of $x_E, x_A, \ldots$. Hence, by (7), the irreducible function $\Phi(x)$ is a factor of $\Theta(x)$.

The remarkable property (5) enjoyed by the group-determinant $\Theta(x)$ and the analogous property (7) enjoyed by an irreducible factor of $\Theta(x)$ play a fundamental rôle in the present theory. These properties emphasize the special significance of the function called a group-determinant among the totality of integral functions of the variables $x_E, x_A, \ldots$

To effect the complete decomposition of the group-determinant $\Theta(x)$, it is found convenient to fix the attention on certain coefficients in the (as yet unknown) irreducible factors of $\Theta(x)$ and call them *group-characters.* This is done for the linear factors in §6 and for general factors in §7. In terms of them all the remaining coefficients of the desired factors will be found to be readily expressible.

6. With the aid of (7), we can determine all the linear factors

$$\Phi(x) \equiv \sum_R \chi(R)\, x_R \qquad (R = E, A, B, \cdots), \tag{8}$$

where the h coefficients are considered to be values of a function χ, the value of $\chi(E)$ being unity. By (7) in connection with (3), we have

$$(\sum_R \chi(R)x_R)(\sum_S \chi(S)y_S) = \sum_T \chi(T)z_T = \sum_{R,\,S} \chi(RS)x_R y_S.$$

Equating the coefficients of $x_R y_S$ in the first and third sums, we get

$$\chi(RS) = \chi(R)\,\chi(S). \tag{9}$$

Thus $\chi(R^2) = [\chi(R)]^2$ and by induction $\chi(R^r) = [\chi(R)]^r$. Hence, if R is of period r, the left member is unity, so that $\chi(R)$ is an rth root of unity. Also $\chi(R)\,\chi(R^{-1}) = \chi(E) = 1$, so that

$$\chi(R^{-1}S^{-1}RS) = \chi(R^{-1})\,\chi(S^{-1})\,\chi(R)\,\chi(S) = 1.$$

Hence $\chi(F) = 1$ for every *commutator* $F = R^{-1}S^{-1}RS$ of the group H. Since

$$T^{-1}FT = (T^{-1}RT)^{-1}(T^{-1}ST)^{-1}(T^{-1}RT)(T^{-1}ST),$$

every conjugate within H of a commutator is itself a commutator. Hence the group G of order g generated by all the commutators of H is an invariant subgroup of H. Let E, A, B, C, . . . be the right-hand multipliers which give rise to a rectangular array for H with the elements of G in the first row. Upon the h/g rows GE, GA, GB, ... as elements, we have a *group*, since $(GA)(GB) = GAB$, in view of the relation $AGA^{-1} = G$. This group is called the *quotient-group* H/G. It is here a *commutative* group. For, if GA and GB are any two of its elements, there exists in the commutator group G the element $F_1 = BAB^{-1}A^{-1}$, for which

$$GBA = GF_1AB = GAB, \qquad (GB)(GA) = (GA)(GB).$$

Since $\chi(F) = 1$, if F is any element of G,

$$\chi(FA) = \chi(F)\,\chi(A) = \chi(A).$$

Hence $\chi(R)$ has the same value for all the elements R of the row GA. Hence there are h/g such values $\chi(E)$, $\chi(A)$, $\chi(B)$, . . . , defined by the respective rows GE, GA, GB, ... Moreover, for F and F' chosen arbitrarily from G, $\chi(FA \cdot F'B)$ has a constant value $\chi(A)\,\chi(B)$. Hence there is defined a function χ whose h/g values for the quotient-group H/G have the property (9). Such a function χ is called a *character* of this commutative group H/G.

Consider, in general, a *commutative* group N of order n. By an elementary, but fundamental, theorem* on commutative groups, a system of elements A_1, A_2, . . . , A_ν of periods a_1, a_2, . . . , a_ν, respectively, can be chosen such that every element A of N can be expressed in one and only one way in the form

$$A = A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_\nu^{\alpha_\nu} \qquad (0 \leqq \alpha_1 < a_1, \cdots, 0 \leqq \alpha_\nu < a_\nu).$$

That these elements A form a commutative group of order $n = a_1 a_2 \cdots a_\nu$, in case A_1, A_2, . . . , A_ν are independent commutative elements, is directly evident. To the elements R, S, . . . of the group N we make correspond any numbers $\chi(R)$, $\chi(S)$, . . . such that property (9) holds and such that $\chi(R)$ is zero for no element R. Setting $S = E$, the identity, we get, from (9), $\chi(R) = \chi(R)\,\chi(E)$, whence $\chi(E) = 1$. It follows, as at the beginning of the section, that the number $\chi(R)$ is an rth root of unity, r being the period of the element R. The function χ is called a *character* of the commutative group N. Two characters χ and χ_1 are said to be different when $\chi(A) \neq \chi_1(A)$

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* Weber, *Algebra*, II, Zweiter Abschnitt; Burnside, *The Theory of Groups*, p. 50.

for at least one element A of N. It is next shown that *the number of different characters equals the order n of the group N.* Set $\chi(A_1) = w_1, \cdots, \chi(A_\nu) = w_\nu$. Then by (9)

$$\chi(A) \equiv \chi(A_1^{a_1}A_2^{a_2}\cdots A_\nu^{a_\nu}) = w_1^{a_1}w_2^{a_2}\cdots w_\nu^{a_\nu}. \tag{10}$$

But $w_1^{a_1} = 1, \cdots, w_\nu^{a_\nu} = 1$, a_1 being the period of $A_1, \cdots, a_\nu$ the period of A_ν. Hence there are at most $n = a_1a_2\cdots a_\nu$ different characters. To show that there are at least n different characters, consider an arbitrary root w_i of $w_i^{a_i} = 1$ for $i = 1, \cdots, \nu$, and define $\chi(A)$ by formula (10). This function χ evidently satisfies the relation (9), while $\chi(A)$ is zero for no element A in the group. Hence χ defines a character. Moreover, a second set of roots $w_1', \ldots, w_\nu'$ leads to a different character χ', since

$$w_1'^{a_1}\cdots w_\nu'^{a_\nu} \equiv w_1^{a_1}\cdots w_\nu^{a_i} \qquad (\text{for every } a_1, \cdots, a_\nu)$$

requires $w_1' = w_1$ (by taking $a_1 = 1, a_2 = 0, \cdots, a_\nu = 0$), $\cdots, w_\nu' = w_\nu$.

Returning to the commutative group H/G of order h/g, let ψ be one of its h/g characters and set $\chi(R) = \psi(GA)$ for every element R of the row GA of the rectangular array. Then the function (8), with the value just assigned to $\chi(R)$, is a factor of the group-determinant $\Theta(x)$. Indeed, if we set $y_R = \chi(R)x_R$, we find that

$$y_{PQ^{-1}} = \chi(PQ^{-1})x_{PQ^{-1}} = \chi(P)\chi(Q^{-1})x_{PQ^{-1}}, \qquad |y_{PQ^{-1}}| = |x_{PQ^{-1}}|.$$

But the first determinant was shown to have the factor Σy_R by adding to its first row the remaining rows. After the removal of this factor, the elements of the first row all equal 1. By subtracting the first column from the remaining columns, we see that the resulting determinant depends only upon the differences $y_R - y_S$ and hence is not divisible by the sum Σy_R. Hence the second determinant $|x|$ has the factor $\Sigma\chi(R)x_R$ and contains it only to the first power. We have, therefore, proved the following

THEOREM. *The linear factors of the group-determinant enter to the first power only. Their number is the quotient of the order of H by the order of its commutator group G.*

EXAMPLE. If H is a cyclic group, it is commutative and the order of G is unity. Then $\Theta(x)$ is a so-called cyclic determinant of order h and has h distinct linear factors.

For the cyclic group H_3, the factors and the corresponding characters are

	$\chi(E)$	$\chi(A)$	$\chi(B)$
$x_E + x_A + x_B$,	1	1	1
$x_E + \rho x_A + \rho^2 x_B$,	1	ρ	ρ^2
$x_E + \rho^2 x_A + \rho x_B$,	1	ρ^2	ρ

where $\rho \equiv -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ is an imaginary cube root of unity.

7. Let Φ be an integral homogeneous function of degree f in the variables $x_E, x_A, x_B, \ldots$, which satisfies condition (7). In it the coefficient of x_E^f is unity. In proof, we take $y_E = 1$, $y_A = y_B = \cdots = 0$, whence $(z) = (x)$, $\Phi(x) = \Phi(x)\,\Phi(1, 0, 0, \ldots)$. Hence $\Phi(1, 0, 0, \ldots)$, the coefficient of x_E^f in $\Phi(x)$, is unity. Denote by $\chi(R)$ the coefficient of x_E^{f-1} in $\dfrac{\partial \Phi}{\partial x_R}$, so that, for $R \neq E$, $\chi(R)$ is the coefficient of $x_E^{f-1} x_R$ in Φ, while

$$\chi(E) = f. \tag{11}$$

The h constants $\chi(R)$ are considered to be values of a function χ. If Φ is irreducible, χ is called the *character of degree f* which corresponds to the factor Φ. As shown in §6, the characters of degree $f = 1$ are roots of unity.

Consider the following expansion, u being a variable,

$$\Phi(x_E + u, x_A, x_B, \cdots) = u^f + \Phi_1 u^{f-1} + \Phi_2 u^{f-2} + \cdots + \Phi_f, \tag{12}$$

where Φ_m is an integral homogeneous function of degree m of $x_E, x_A, x_B, \ldots$ In particular, $\Phi_f = \Phi$ and

$$\Phi_1 = \sum_R \chi(R)\, x_R \qquad (R = E, A, B, \ldots). \tag{13}$$

If the right member of (12) vanishes for $u = -u_1, -u_2, \cdots, -u_f$, then

$$\Phi(x_E + u, x_A, x_B, \cdots) \equiv (u + u_1)(u + u_2) \cdots (u + u_f). \tag{14}$$

In an analogous manner, we set

$$u^n + v \equiv (u + v_1)(u + v_2) \cdots (u + v_n). \tag{15}$$

If the group-matrix (x) be compounded with itself n times, there results, as in §2, a matrix $(x)^n = (x^{(n)}_{PQ^{-1}}) \equiv (x^{(n)})$, where

$$x^{(n)}_R = \sum_{R_1, \ldots, R_n} x_{R_1} x_{R_2} \cdots x_{R_n} \qquad (R_1 R_2 \cdots R_n = R). \tag{16}$$

Let (ϵ) denote the unit-matrix, whose elements are all zero except those in the main diagonal which are unity. Then from (15) follows*

$$(x)^n + v(\epsilon) \equiv (x^{(n)}) + v(\epsilon) \equiv [(x) + v_1(\epsilon)] \cdot \cdots \cdot [(x) + v_n(\epsilon)].$$

Since the matrices $[(x) + v_i(\epsilon)]$ possess the same symmetry as the group matrix (x) we may apply to the former the relation (7). Hence

$$\Phi(x_E^{(n)} + v,\, x_A^{(n)},\, x_B^{(n)},\, \cdots) = \prod_{i=1}^{n} \Phi(x_E + v_i,\, x_A,\, x_B,\, \cdots)$$

$$= \prod_{i=1}^{n} (v_i + u_1)(v_i + u_2) \cdots (v_i + u_f) = (v + u_1^n)(v + u_2^n) \cdots (v + u_f^n),$$

upon applying (14) and (15). The coefficient of v^{f-1} in the last product is

$$u_1^n + u_2^n + \cdots + u_f^n \equiv S_n$$

In view of (12) and (13), the coefficient of v^{f-1} in $\Phi(x_E^{(n)} + v, x_A^{(n)}, \ldots)$ is $\Sigma\chi(R)x_R^{(n)}$. Hence

$$S_n = \sum_R \chi(R) x_R^{(n)} \qquad (R = E, A, B, \cdots). \tag{17}$$

In view of (16), this result may be written

$$S_n = \sum_{R_1, \ldots, R_n} \chi(R_1 R_2 \cdots R_n) x_{R_1} x_{R_2} \cdots x_{R_n} \quad (R_i = E, A, B, \cdots). \tag{18}$$

The coefficients $\Phi_1, \Phi_2, \ldots$ of (12), in particular the desired function $\Phi_f \equiv \Phi$, can be expressed in terms of the sums of powers $S_1, S_2, \ldots, S_n, \ldots$ of the roots with their signs changed (compare Serret, *Algèbre supérieure*, 4th ed., vol. 1, p. 460). For example

$$\begin{aligned} &2\Phi_2 = S_1^2 - S_2, \quad 6\Phi_3 = S_1^3 - 3S_1S_2 + 2S_3, \\ &24\Phi_4 = S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4. \end{aligned} \tag{19}$$

Properties of Group-Characters.

8. For $y_A = 1$ and the remaining $y_S = 0$, (3) gives $z_T = x_{TA^{-1}}$, and $\Phi(y)$ reduces to a constant $\psi(A)$. Then relation (7) becomes

$$\Phi(x_{TA^{-1}}) = \psi(A)\Phi(x_T). \tag{20}$$

For $x_A = 1$, $x_R = 0$ $(R \neq A)$, (3) gives $z_T = y_{A^{-1}T}$. Then (7) becomes

$$\Phi(y_{A^{-1}T}) = \psi(A)\Phi(y_T). \tag{21}$$

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* By a principle of the theory of matrices, an identity between two integral rational functions of a variable u leads to an analogous identity obtained by replacing u by a matrix (x) and the constant term $C \equiv Cu^0$ by $C(x)^0 \equiv C(\epsilon)$.

Replacing T by AT in (21), and then A by A^{-1}, we get

$$\Phi(y_T) = \psi(A)\Phi(y_{AT}) = \psi(A)\ \psi(A^{-1})\ \Phi(y_T).$$

Hence $\psi(A)\ \psi(A^{-1}) = 1$. Replacing y_T by x_{TA} in (21), and using (20) we get

$$\Phi(x_{A^{-1}TA}) = \psi(A)\ \Phi(x_{TA}) = \psi(A)\ \psi(A^{-1})\ \Phi(x_T),$$

$$\Phi(x_{A^{-1}TA}) = \Phi(x_T). \tag{22}$$

Equating the coefficients of $x_E^{f-1}x_R$, we get $\chi(ARA^{-1}) = \chi(R)$. *Hence a character has the same value for all elements conjugate within H.* Setting $R = BA$, we get

$$\chi(AB) = \chi(BA) \qquad (A,\ B \text{ any elements of } H). \tag{23}$$

From the usual relations between the elements of a determinant Θ and its adjoint minors, we have

$$\sum_R x_{PR^{-1}}\Theta_{QR^{-1}} = \sum_R x_{RP^{-1}}\Theta_{RQ^{-1}} \equiv \begin{cases} \Theta \text{ if } P = Q, \\ 0 \text{ if } P \neq Q. \end{cases}$$

Setting $T = QR^{-1}$, $A = PQ^{-1}$ in the first sum, but $T = RQ^{-1}$, $A = QP^{-1}$ in the second sum, and applying formula (2), we get

$$\sum_T x_{AT}\frac{\partial\Theta}{\partial x_T} = \sum_T x_{TA}\ \frac{\partial\Theta}{\partial x_T} = \epsilon_A h\Theta, \tag{24}$$

where $\epsilon_A = 1$ if $A = E$, the identity, while $\epsilon_A = 0$ if $A \neq E$. In view of (6), $\Theta = \Phi^e\Psi$, where Φ and Ψ have no common factor. Hence

$$\sum_T x_{AT}\left(e\Psi\frac{\partial\Phi}{\partial x_T} + \Phi\frac{\partial\Psi}{\partial x_T}\right) = \epsilon_A h\Phi\Psi.$$

Hence $\Sigma x_{AT}\dfrac{\partial\Phi}{\partial x_T}$ is divisible by Φ, and the quotient is a constant, being of degree zero. The latter is $\chi(A^{-1})$, as shown by comparing the coefficients of x_E^f. Hence*

$$\sum_T x_{AT}\frac{\partial\Phi}{\partial x_T} = \sum_T x_{TA}\ \frac{\partial\Phi}{\partial x_T} = \chi(A^{-1})\Phi. \tag{25}$$

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* Another proof is given by differentiating the terms of (7) with respect to y_A and setting $y_R = \epsilon_R$ in the result (and with respect to x_A and setting $x_R = \epsilon_R$), applying (3) in each case.

Multiplying the first and third members by $h\Theta_{AS}$ and summing with respect to A,

$$h\sum_T\Big(\sum_A x_{AT}\Theta_{AS}\Big)\frac{\partial\Phi}{\partial x_T} \equiv h\Theta\frac{\partial\Phi}{\partial x_S} = \Phi\sum_A\chi(A^{-1})\frac{\partial\Theta}{\partial x_{AS}}.$$

Setting first $\Theta = \Phi^e\Psi$ and replacing AS by R in the last member, we get

$$h\Psi\frac{\partial\Phi}{\partial x_S} = \sum_R\chi(SR^{-1})\left(e\Psi\frac{\partial\Phi}{\partial x_R} + \Phi\frac{\partial\Psi}{\partial x_R}\right).$$

The terms which multiply Ψ are equal, since their difference is divisible by Φ, while their degree is $f-1$. Hence

$$\frac{h}{e}\frac{\partial\Phi}{\partial x_S} = \sum_R\chi(SR^{-1})\frac{\partial\Phi}{\partial x_R}. \tag{26}$$

Setting next $\Theta = \Phi'^{e'}\Psi'$, Φ' being an irreducible factor distinct from Φ, we get

$$h\Phi'\Psi'\frac{\partial\Phi}{\partial x_S} = \Phi\sum_R\chi(SR^{-1})\left(e'\Psi'\frac{\partial\Phi'}{\partial x_R} + \Phi'\frac{\partial\Psi'}{\partial x_R}\right).$$

Hence the following sum is divisible by Φ' and therefore vanishes:

$$\sum_R\chi(SR^{-1})\frac{\partial\Phi'}{\partial x_R} = 0. \tag{27}$$

Comparing the coefficients of x_E^{f-1} in (26) and those of $x_E^{f'-1}$ in (27), we get

$$\sum_R\chi(SR^{-1})\chi(R) = \frac{h}{e}\chi(S), \qquad \sum_R\chi(SR^{-1})\psi(R) = 0, \tag{28}$$

where $\psi(R)$ is the character corresponding to the factor Φ'. For $S = E$,

$$\sum_R\chi(R)\chi(R^{-1}) = \frac{hf}{e}, \qquad \sum_R\psi(R)\chi(R^{-1}) = 0. \tag{29}$$

Hence the values $\chi(E), \chi(A), \ldots$ are not proportional to $\psi(E), \psi(A), \ldots$ Replacing R by RB and S by AB in (28),

$$\sum_R\chi(AR^{-1})\,\chi(BR) = \frac{h}{e}\chi(AB). \tag{28'}$$

Another important property of characters is derived in §10.

9. The *special group-matrix* (η) is obtained from (y) by setting

$$\eta_R \equiv y_R = y_{A^{-1}RA} = y_{B^{-1}RB} = y_{C^{-1}RC} = \cdots, \tag{30}$$

so that the number of distinct elements η_R equals the number r of sets o conjugate operators of the group H. The determinant $|\eta|$ is called the *special group-determinant.** Then (η) is commutative with any matrix (x) of the form (1). Indeed, by §2 $(x)(y)$ is of the form (z), where

$$z_{PQ^{-1}} = \sum_S x_{PQ^{-1}S^{-1}} y_S \equiv \sum_S x_S y_{S^{-1}PQ^{-1}},$$

upon replacing S by $S^{-1}PQ^{-1}$ in the second sum. Also $(y)(x)$ is of the form (z'), where

$$z'_{PQ^{-1}} = \sum_S y_{PQ^{-1}S^{-1}} x_S = \sum_S y_{S^{-1}(PQ^{-1}S^{-1})S}\, x_S = z_{PQ^{-1}}.$$

Hence an arbitrary group-matrix $(x_{PQ^{-1}})$, a special group-matrix $(y_{PQ^{-1}})$, and $(\epsilon_{PQ^{-1}})$, the unit-matrix, are commutative with each other. By a theorem of Frobenius,† the determinant

$$|u x_{PQ^{-1}} + v y_{PQ^{-1}} + w \epsilon_{PQ^{-1}}|$$

is a product of linear homogeneous functions of the variables u, v, w. Hence

$$(31)\quad \Phi(u x_E + v y_E + w,\ u x_A + v y_A,\ u x_B + v y_B,\ \cdots) = \prod_{i=1}^{f} (u_i u + v_i v + w),$$

the coefficients of w being chosen to be unity in view of the term w^f on the left. Setting $u = 1$, $v = 0$, we obtain (compare (14) of §7)

$$(32)\quad \Phi(x_E + w,\ x_A,\ x_B,\ \cdots) = (u_1 + w)(u_2 + w)\cdots(u_f + w),$$

so that $u_1, u_2, \ldots, u_f$ depend only upon the variables $x_E, x_A, \ldots$ Similarly, $v_1, v_2, ., v_f$ depend only upon $y_E, y_A, \ldots$ Since $\Phi(x)$ is irreducible, the function (32) is irreducible, considered as a function of w, *i.e.*, it is not expressible as a product of two integral rational functions of w whose coefficients are rational functions of $x_E, x_A, \ldots.$ Replacing w by $w + v_1$ in (32), we obtain the irreducible function of w,

$$\Phi(x_E + v_1 + w,\ x_A,\ x_B,\ \cdots) = (u_1 + v_1 + w)(u_2 + v_1 + w)\cdots(u_f + v_1 + w).$$

It has the factor $u_1 + v_1 + w$ in common with $\Phi(x_E + y_E + w,\ x_A + y_A,\ \cdots)$, obtained by setting $u = v = 1$ in (31). Hence the two functions are identical. Taking $w = 0$, we get

$$(33)\quad \Phi(x_E + y_E,\ x_A + y_A,\ x_B + y_B,\ \cdots) = \Phi(x_E + v_1,\ x_A,\ x_B,\ \cdots),$$

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* Inversely, a matrix (y) with elements in a field F and commutative with every matrix (x) in F is a special matrix (η), *Transactions, l. c.*, §2.

† Über vertanschbare Matrizen, *Berliner Sitzungsberichte*, 1896, p. 602,

v_1 being a function of $y_E, y_A, \cdots$ Setting $x_E = u$, $x_A = x_B = \cdots = 0$, we get

$$\Phi(y_E + u, y_A, y_B, \cdots) = (u + v_1)^f. \tag{34}$$

Equating the coefficients of u^{f-1} and applying the definitions of §7, we get

$$\sum_R \chi(R) y_R = f v_1. \tag{35}$$

An irreducible factor Φ of degree f of the general group-determinant becomes, upon imposing the conditions (30), *the fth power of a linear factor $\frac{1}{f}\Sigma\chi(R)y_R$ of the special group-determinant.*

Since the values of the character $\psi(R)$ corresponding to an irreducible factor Φ' distinct from Φ are not proportional to the values of $\chi(R)$ corresponding to Φ (end of §8), the linear function $\frac{1}{f'}\Sigma\psi(R)y_R$ is not a constant times the linear function $\frac{1}{f}\Sigma\chi(R)y_R$.

Two distinct irreducible factors of the general group-determinant give rise to two independent linear factors of the special group-determinant.

10. Let the elements $E, A, B, \ldots$ of the group H be separated into distinct sets of conjugates and designate the sets as the 1st, 2d, ..., ρth, ..., rth set. Let h_ρ denote the number of elements R in the ρth set; for each of them $\chi(R)$ has the same value χ_ρ. In view of (30), there are r distinct variables $y_1, y_2, \ldots, y_r$. Then (35) becomes

$$\sum_{\rho=1}^{r} h_\rho \chi_\rho y_\rho = f v_1. \tag{35'}$$

Differentiating (33) with respect to y_ρ and setting $y_1 = \cdots = y_r = 0$ in the result, we get

$$\sum_R^{\rho\text{th set}} \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_R} = \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_E}\,\frac{\partial v_1}{\partial y_\rho} = \frac{1}{f} h_\rho \chi_\rho \,\frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_E}.$$

Let A be a fixed element and replace each x_R by x_{AR}. By (21), $\Phi(x_T)$ is thereby multiplied by a constant $\psi(A^{-1})$. It follows that

$$\sum_R^{\rho\text{th set}} \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_{AR}} = \frac{1}{f} h_\rho \chi_\rho \,\frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_A}.$$

Equating the coefficients of x_E^{f-1}, and replacing χ_ρ by $\chi(B)$, we get

$$(36) \qquad \sum_R^{\rho\text{th set}} \chi(AR) = \frac{1}{f} h_B \chi(B)\chi(A),$$

where R runs through h_B elements conjugate with B. Set $R = S^{-1}BS$, and let S run through the series of h elements of H, so that R becomes identical h/h_B times with each element conjugate with B. Hence the left member of (36) equals $\frac{1}{h} h_B \sum_S \chi(AS^{-1}BS)$, so that

$$(37) \qquad h\chi(A)\chi(B) = f \sum_R \chi(AR^{-1}BR).$$

11. Let l denote the number of distinct irreducible factors $\Phi, \Phi', \ldots$ of Θ. Corresponding to $\Phi^{(\lambda)}$, which occurs to the power $e^{(\lambda)}$ in Θ, there exists a character $\chi^{(\lambda)}$, for which $\chi^{(\lambda)}(E) = f^{(\lambda)}$, the degree of $\Phi^{(\lambda)}$. Since Θ is of degree h,

$$(38) \qquad \sum_\lambda e^{(\lambda)} \chi^{(\lambda)}(E) = \sum_\lambda e^{(\lambda)} f^{(\lambda)} = h.$$

By the definition in §7, $\chi^{(\lambda)}(R)$ is the coefficient of $x^{f^{(\lambda)}-1}x_R$ in $\Phi^{(\lambda)}$, if $R \neq E$; while the coefficient of $x_E^{f^{(\lambda)}}$ is 1. Hence the coefficient of $x_E^{h-1}x_R$ in the product (6) is $\Sigma e^{(\lambda)}\chi^{(\lambda)}(R)$. But x_E occurs only in the main diagonal of $\Theta(x)$, so that the coefficient of $x_E^{h-1}x_R$ is zero if $R \neq E$. Hence

$$(39) \qquad \sum_\lambda e^{(\lambda)} \chi^{(\lambda)}(R) = 0. \qquad (R \neq E).$$

By (38) and (39), we have

$$\sum_\lambda \frac{1}{h} e^{(\lambda)} \chi^{(\lambda)}(B^{-1}R^{-1}BR) = \begin{cases} 1 & \text{if } RB = BR, \\ 0 & \text{if } RB \neq BR. \end{cases}$$

Now the number of pairs of elements R, B of H for which $RB = BR$ is rh, where r is the number of sets of conjugates in H. Indeed, if R be a definite element of the ρth set there exist h/h_ρ elements B commutative with R. If R runs through all h_ρ elements of the ρth set, there results $h_\rho \cdot h/h_\rho$ pairs of solutions of $RB = BR$. This number h being the same for each of the r sets, there results hr pairs of solutions in all. Hence the sum (in which B, R run through the series of all elements of H)

$$(40) \qquad \sum_{B,R} \sum_\lambda \frac{1}{h} e^{(\lambda)} \chi^{(\lambda)}(B^{-1}R^{-1}BR)$$

equals hr. But, by (37) and (29), we have

$$\sum_B \left[\frac{e}{h} \sum_R \chi(B^{-1}R^{-1}BR)\right] = \frac{e}{f} \sum_B \chi(B^{-1})\chi(B) = h.$$

Since the first sum equals h for each of the l values of λ, the sum (40) equals hl. Hence $l = r$, which proves the following theorem:

The number of distinct irreducible factors of the group-determinant Θ *equals the number* r *of sets of conjugate elements in the group* H.

Corollary. The number of distinct linear factors of the special group-determinant equals the number r of sets of conjugate elements.

12. As an example, consider the symmetric group on 3 letters and set

$$E = \text{identity},\ A = (132),\ B = (123),\ C = (12),\ D = (13),\ F = (23).$$

The group-matrix (1) then becomes

$$(41)\qquad \begin{pmatrix} x_E & x_B & x_A & x_C & x_D & x_F \\ x_A & x_E & x_B & x_D & x_F & x_C \\ x_B & x_A & x_E & x_F & x_C & x_D \\ x_C & x_D & x_F & x_E & x_B & x_A \\ x_D & x_F & x_C & x_A & x_E & x_B \\ x_F & x_C & x_D & x_B & x_A & x_E \end{pmatrix}$$

The commutator group (§6) is an invariant subgroup and contains $C^{-1}B^{-1}CB = A$; hence it is the cyclic group $\{E, A, B\}$. Hence there are exactly 6/3 characters of the first degree. If χ be one of these two characters, we have

$$(42)\qquad \chi(E) = \chi(A) = \chi(B) = 1. \qquad \chi(C) = \chi(D) = \chi(F) = \pm 1.$$

Hence the linear factors of the group-determinant Θ are

$$(43)\qquad x_E + x_B + x_A + x_C + x_D + x_F, \qquad x_E + x_B + x_A - x_C - x_D - x_F.$$

Since the number of sets of conjugate elements is $r = 3$, there is only one irreducible factor Φ of degree $f > 1$. It will be determined by a general method in §15. In the present simple case, it may be determined by inspection as follows. Let ω be an imaginary cube root of unity and set

$$x = x_E + \omega x_A + \omega^2 x_B,\ y = x_E + \omega^2 x_A + \omega x_B,\ z = x_C + \omega x_D + \omega^2 x_F,$$
$$w = x_C + \omega^2 x_D + \omega x_F.$$

Multiply the second row of Θ by ω and the third row by ω^2 and add the results to the first row; multiply the fifth row by ω^2 and the sixth row by ω

and add the results to the fourth row. The new first and fourth rows are

$$\begin{matrix} x & \omega x & \omega^2 x & z & \omega^2 z & \omega z, \\ w & \omega w & \omega^2 w & y & \omega^2 y & \omega y. \end{matrix}$$

In Laplace's development of the new determinant, each term is the product of a determinant of order two formed from the above two rows by the complementary determinant of order four. Since each of the former has the factor $xy - zw \equiv \Phi$, the determinant Θ has the factor Φ. Since x, y, z, w are independent functions of x_E, x_A, . . . , Φ is irreducible. The value of Φ is

$$(44) \quad x_E^2 + x_A^2 + x_B^2 - x_E x_A - x_E x_B - x_A x_B \\ - x_C^2 - x_D^2 - x_F^2 + x_C x_D + x_C x_F + x_D x_F.$$

Hence* Φ must occur in Θ to the power $e = 2$, so that $f = e = 2$. By §7, the corresponding character $\chi^{(2)}$ has the values

$$(45) \quad \begin{gathered} \chi^{(2)}(E) = 2, \quad \chi^{(2)}(A) = \chi^{(2)}(B) = -1, \\ \chi^{(2)}(C) = \chi^{(2)}(D) = \chi^{(2)}(F) = 0. \end{gathered}$$

EQUALITY OF THE NUMBERS e AND f.

13. The most difficult part of the whole theory, as Frobenius states, is the proof that the degree f of an irreducible factor Φ of the group-determinant Θ equals the exponent e of the power to which Φ enters Θ.

For $f = 1$, the exponent e is 1 by §6.

For $f = 2$, relations (18) and (19) give

$$(46) \quad 2\Phi = S_1^2 - S_2 = \sum_{A,B} [\chi(A)\,\chi(B) - \chi(AB)]\, x_A x_B.$$

Now Φ_3, Φ_4, . . . do not occur in (12); but, if they be *defined* by (19), they are all zero. That Φ_3, Φ_4, . . . are identically zero, may be verified directly. Thus $6\Phi_3$ equals

$$S_1^3 - 3S_1S_2 + 2S_3 \equiv (u_1 + u_2)^3 - 3(u_1 + u_2)(u_1^2 + u_2^2) + 2(u_1^3 + u_2^3) \equiv 0.$$

In this we substitute the value of S_n given by (18) and determine the coefficient of $x_A x_B x_C$. If A, B, C are distinct there occurs a common factor

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* In a note in the *American Mathematical Monthly*, March, 1902, I prove these results by the most elementary principles of determinants. They also follow by methods of group theory (Dedekind, *Berliner Sitzungsberichte*, 1897, p. 1007; Dickson, *Trans. Amer. Math. Soc.*, vol. 3 (1902), p. 296).

6; if two are equal, the factor 3; if all are equal, the factor 1. Cancelling the factor, we get

$$\chi(A)\,\chi(B)\,\chi(C) - \chi(A)\,\chi(BC) - \chi(B)\,\chi(AC) - \chi(C)\,\chi(AB) + \chi(ABC) + \chi(ACB) = 0.$$

Replacing B by BC^{-1} and summing over all h elements C of H, we get

$$\chi(A) \sum_{C} \chi(BC^{-1})\,\chi(C) - h\chi(A)\,\chi(B) - \Sigma\, \chi(BC^{-1})\,\chi(AC) - \Sigma\, \chi(C)\,\chi(ABC^{-1}) + h\chi(AB) + \Sigma\, \chi(ACBC^{-1}) = 0.$$

Applying relations (23), (28′), and (37), we get

$$\frac{h}{e}\,\chi(A)\,\chi(B) - h\chi(A)\,\chi(B) - \frac{h}{e}\,\chi(AB) - \frac{h}{e}\,\chi(AB) + h\chi(AB) + \frac{h}{f}\,\chi(A)\,\chi(B) = 0.$$

Since $f = 2$, this relation factors into

$$\left(\frac{h}{e} - \frac{h}{2}\right)\left[\chi(A)\,\chi(B) - 2\chi(AB)\right] = 0.$$

If the second factor vanished, we should obtain, upon multiplying it by $x_A x_B$ and summing over all pairs of elements A, B of the group, the result $S_1^2 - 2S_2 = 0$. Then would $4\Phi = S_1^2$, whereas Φ is irreducible.* Hence must $e = 2 = f$.

For $f = 3$, we have by (19),

$$6\Phi = S_1^3 - 3S_1S_2 + 2S_3, \qquad 24\Phi_4 = S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4 \equiv 0. \tag{47}$$

For brevity set (R) for $\chi(R)$. The coefficient of $x_A x_B x_C x_D$ in $24\Phi_4$ is (aside from a factor 24, 12, or 4):

$$\begin{aligned}&(A)(B)(C)(D) - (B)(C)(AD) - (A)(C)(BD) - (A)(B)(CD)\\ &- (A)(D)(BC) - (B)(D)(AC) - (C)(D)(AB) + (A)(BCD)\\ &+ (B)(ACD) + (C)(ABD) + (D)(ABC) + (A)(BDC) + (B)(ADC)\\ &+ (C)(ADB) + (D)(ACB) + (BC)(AD) + (AC)(BD) + (AB)(CD)\\ &- (ABCD) - (ACBD) - (BACD) - (BCAD) - (CABD) - (CBAD)\\ &= 0.\end{aligned}$$

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* To give another proof, like that used for the case of $f = 4$, we replace A by AB^{-1} and sum for B. We obtain $\sum_{B}[\chi(AB^{-1})\chi(B) - 2\chi(A)] \equiv \chi(A)(h/e - 2h)$, upon applying (28). But the last expression is not zero.

Replacing C by CD^{-1} and summing over all h elements D, and applying (23) (28′), (37), we get

$$\frac{h}{e}\,[(A)(B)(C)-2(B)(AC)-2(A)(BC)-(C)(AB)+3(ABC)$$
$$+3(ACB)]+h\,[-(A)(B)(C)+(C)(AB)+(A)(BC)+B(AC)$$
$$-(ABC)-(ACB)]+\frac{h}{f}\,[2(A)(B)(C)-(B)(AC)-(A)(BC)$$
$$-2(C)(AB)]=0.$$

Since $f=3$, this relation factors into

$$\left(\frac{h}{e}-\frac{h}{3}\right)[(A)(B)(C)-2(A)(BC)-2(B)(AC)-(C)(AB)$$
$$+3(ABC)+3(ACB)]=0\cdot$$

If the second factor vanished identically, we should obtain, upon multiplying it by $x_A x_B x_C$ and summing over all triples A, B, C, the result $S_1^3-5S_1S_2+6S_3=0$, so that, by (47), $9\Phi=S_1(S_1^2-2S_2)$, contrary to the irreducibility of Φ. Hence $e=3=f$.

For $f=4$, we have $\Phi=\Phi_4$, defined by (47). Also

$$5\,!\,\Phi_5=S_1^5-10S_1^3S_2+20S_1^2S_3+15S_1S_2^2-30S_1S_4-20S_2S_3$$
$$+24S_5\equiv 0.$$

For the coefficient of $x_A x_B x_C x_D x_F$ in $5\,!\Phi_5$, apart from a common numerical factor, we have 120 terms of which the first is $(A)(B)(C)(D)(F)$, while 10 are of the type $-(A)(B)(C)(DF)$, 20 of the type $(A)(B)(CDF)$, 15 of the type $(A)(BC)(DF)$, 30 of the type $-(A)(BCDF)$, 20 of the type $-(AB)(CDF)$, and 24 of the type $(ABCDF)$. Each term corresponds to a substitution on five letters, its sign being $+$ or $-$ according as the substitution is even or odd. We therefore take all permutations of the letters in any type, aside from cyclic interchanges within the cycles, which leave the character unaltered by (23). Replacing D by DF^{-1} and summing over all h elements F, and applying (23), (28′), (37), we get

$$\text{(48)}\qquad \frac{h}{e}\,U+h\,V+\frac{h}{f}\,W=0,$$

$$U\equiv(A)(B)(C)(D)-2(A)(C)(BD)-2(B)(C)(AD)$$
$$-2(A)(B)(CD)+(D)(BAC)+(D)(ABC)-(C)(D)(AB)$$
$$-(A)(D)(BC)-(B)(D)(AC)+2(AB)(CD)+2(AC)(BD)$$
$$+2(AD)(BC)+3(A)(BCD)+3(A)(CBD)+3(B)(ACD)$$

$$\begin{aligned}&+3(B)(CAD)+3(C)(ABD)+3(C)(BAD)-4(ABCD)\\&-4(ABDC)-4(ACBD)-4(ACDB)-4(ADBC)-4(ADCB).\end{aligned}$$

$$\begin{aligned}V\equiv &-(A)(B)(C)(D)+(A)(B)(CD)+(A)(C)(BD)+(B)(C)(AD)\\&+(C)(D)(AB)+(A)(D)(BC)+(B)(D)(AC)-(A)(BCD)\\&-(A)(CBD)-(B)(ACD)-(B)(ADC)-(C)(ABD)\\&-(C)(ADB)-(D)(ABC)-(D)(ACB)-(AB)(CD)\\&-(AC)(BD)-(AD)(BC)+(ABCD)+(ABDC)+(ACBD)\\&+(ACDB)+(ADBC)+(ADCB).\end{aligned}$$

$$\begin{aligned}W\equiv\ &3(A)(B)(C)(D)-2(A)(B)(CD)-2(A)(C)(BD)\\&-2(B)(C)(AD)-3(A)(D)(BC)-3(B)(D)(AC)-3(C)(D)(AB)\\&+2(AB)(CD)+2(AC)(BD)+2(AD)(BC)+3(D)(ABC)\\&+3(D)(ACB)+(A)(BCD)+(A)(CBD)+(B)(ACD)\\&+(B)(CAD)+(C)(ABD)+(C)(BAD).\end{aligned}$$

Since $f=4$, the left member of (48) factors into $h/e-h/4$ and U. But if $U\equiv 0$,

$$\sum_{A,B,C,D} Ux_Ax_Bx_Cx_D\equiv S_1^4-9S_1^2S_2+6S_2^2+20S_1S_3-24S_4=0.^*$$

Then, Φ being the Φ_4 of formulae (47), we would have, upon eliminating S_4,

$$96\,\Phi=3S_1^4-15S_1^2S_2+6S_2^2+12S_1S_3. \tag{49}$$

As this expression does not factor, the proof proceeds differently from that for the cases $f=2, f=3$. We replace C by CD^{-1} in U and sum for all h elements D. Then

$$\begin{aligned}&\frac{h}{e}[(A)(B)(C)-3(A)(BC)-3(B)(AC)-(C)(AB)+6(ABC)\\&+6(ACB)]+h[-2(A)(B)(C)+3(A)(BC)+3(B)(AC)\\&+2(C)(AB)-4(ABC)-4(ACB)]+\frac{h}{f}[6(A)(B)(C)-4(A)(BC)\\&-4(B)(AC)-8(C)(AB)]=0.\end{aligned}$$

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* In terms of the roots u_1, u_2, u_3, u_4, of which S_n is the sum of the nth powers, this equation gives $-\Sigma u_i^4+\Sigma u_i^3u_j-\Sigma u_i^2\ u_ju_k+4u_1u_2u_3u_4=0,\qquad \Phi=u_1u_2u_3u_4$.

Reducing, and dropping the common factor h, we get

$$(50)\quad \left(\frac{1}{e}-\frac{1}{2}\right)(A)(B)(C)+\left(2-\frac{3}{e}\right)[(A)(BC)+(B)(AC)] \\ -\frac{1}{e}(C)(AB)+\left(\frac{6}{e}-4\right)[(ABC)+(ACB)]=0.$$

Replacing B by BC^{-1} in (49) and summing over all h elements C, we get

$$h\left(\frac{1}{e^2}-\frac{2}{e}+1\right)[(A)(B)-4(AB)]=0.$$

If the last factor vanished identically, we should find, on replacing A by AB^{-1} and summing over all elements B, that $(he^{-1}-4h)(A)=0$, which is impossible. Hence the second factor vanishes, so that $e=1$. Then (50) becomes

$$\tfrac{1}{2}(A)(B)(C)-(A)(BC)-(B)(AC)-(C)(AB)+2(ABC) \\ +2(ACB)=0.$$

Multiplying by $2x_A x_B x_C$ and summing over all sets A, B, C, we get

$$S_1^3-6S_1S_2+8S_3=0.$$

Eliminating S_3 from (49), we get $192\Phi=3(S_1^2-2S_2)^2$, contrary to the irreducibility of Φ. Hence $U\neq 0$, so that $e=4=f$.

By a quite similar process, involving however general notations, we obtain, for any f, the alternative $e=f$ or $U=0$, where U is a polynomial in $S_1, S_2, \ldots, S_f$, linear in S_f, and with constant coefficients not all zero. Frobenius shows that such a relation $U=0$ is impossible, since it would lead to an equation of degree f in a certain matrix, whereas the equation of lowest degree satisfied by the latter is of degree $f+1$. Instead of this method of greater theoretical difficulty, we may proceed by the elementary method used above for $f=4$.

CALCULATION OF GROUP-CHARACTERS AND GROUP-DETERMINANTS.

14. For $e=f$, equation (23), the first equation (29), and (37), may be written

$$(51)\quad \chi(AB)=\chi(BA),\qquad \sum_R \chi(R)\,\chi(R^{-1})=h, \\ h\chi(A)\,\chi(B)=f\sum_R \chi(AR^{-1}BR).$$

Frobenius in his first paper, "Über Gruppencharaktere," and Weber in his *Algebra*, take the relations (51), with the added condition $f > 0$, as the definition of the general character χ of the group H of order h. The third relation, for $B = E$, gives $\chi(e) = f$. This f is called the degree of the character χ. Starting with these definitions, Frobenius shows that f is always a positive integer and that the number of distinct characters is the number r of sets of conjugate elements of the group H. From the standpoint of this paper, these results have been established in §§7, 11. Whether viewed as definitional equations or as derived relations, equations (51) serve to calculate the r distinct characters. We consider some simple examples.

15. Consider the symmetric group on 3 letters and retain the notations of §12. The three sets of conjugate elements are E; A, B; C, D, F. By the first relation (51), $\chi(A) = \chi(B)$, $\chi(C) = \chi(D) = \chi(F)$. By the second and third relations (51), we get

$$6 = f^2 + 3\chi^2(C) + 2\chi^2(A), \quad 6\chi^2(C) = 2f^2 + 4f\chi(A),$$
$$6\chi^2(A) = 3f^2 + 3f\chi(A), \quad 6\chi(A)\,\chi(C) = 6f\chi(C).$$

If $\chi(C) \neq 0$, the last relation gives $\chi(A) = f$, so that the others give

$$\chi^2(C) = f^2, \quad f^2 = 1.$$

If $\chi(C) = 0$, the second relation gives $\chi(A) = -\frac{1}{2}f$, so that the others give $f^2 = 4$. Since $f > 0$, the results for the two cases are as follows:

f	$\chi(E)$	$\chi(A)$	$\chi(C)$
1	1	1	1
1	1	1	-1
2	2	-1	0

(These results agree with the results (42) and (45) of §12). The two characters of degree $f = 1$ lead, in view of formula (8), to the two linear factors (43). To obtain the irreducible factor Φ_2 of degree $f = 2$, we proceed as in §7. Then

$$S_1 = \sum_R \chi(R)x_R = 2x_E - x_A - x_B,$$
$$\begin{aligned} S_2 = \sum_{R_1, R_2} \chi(R_1R_2)x_{R_1}x_{R_2} = 2(x_E^2 + 2x_Ax_B + x_C^2 + x_D^2 + x_F^2) \\ - (2x_Ex_A + x_B^2 + x_Dx_C + x_Fx_D + x_Cx_F) \\ - (2x_Ex_B + x_A^2 + x_Fx_C + x_Cx_D + x_Dx_F), \end{aligned}$$

the three expressions in parentheses corresponding to the terms of the sum for which $R_1R_2 = E, A, B$, respectively. Then $2\Phi_2 = S_1^2 - S_2$ leads to the result (44).

16. Consider the dihedron group H_{10} generated by a, β subject to

$$a^5 = E, \quad \beta^2 = E, \quad a\beta = \beta a^{-1} \qquad (E = \text{identity}). \tag{52}$$

The 4 distinct sets of conjugates are E; a, a^4; a^2, a^3; $\beta, \beta a, \beta a^2, \beta a^3, \beta a^4$. The 4 corresponding characters are designated $\chi(E) = f$, $\chi(a) = \chi_1$, $\chi(a^2) = \chi_2$, $\chi(\beta) = \chi_3$. According as $R = a^i$ or $R = \beta a^i$, we have

$$aR^{-1}aR = \begin{cases} a^2 \\ E \end{cases}, \quad a^2R^{-1}a^2R = \begin{cases} a^4 \\ E \end{cases}, \quad \beta R^{-1}\beta R = \begin{cases} a^{2i} \\ a^{2i} \end{cases}, \quad aR^{-1}a^2R = \begin{cases} a^3 \\ a^4 \end{cases},$$

$$aR^{-1}\beta R = \begin{cases} \beta a^{2i-1} \\ \beta a^{2i-1} \end{cases}.$$

Hence the second and third relations (51) give

$$10 = f^2 + 2\chi_1^2 + 2\chi_2^2 + 5\chi_3^2, \quad 10\chi_1^2 = 5f^2 + 5f\chi_2, \quad 10\chi_2^2 = 5f^2 + 5f\chi_1,$$

$$10\chi_3^2 = 2f^2 + 4f\chi_1 + 4f\chi_2, \quad 10\chi_1\chi_2 = 5f\chi_1 + 5f\chi_2, \quad 10\chi_1\chi_3 = 10f\chi_3,$$

$$10\chi_2\chi_3 = 10f\chi_3.$$

If $\chi_3 \neq 0$, the last two relations give $\chi_1 = \chi_2 = f$ and the others then give

$$\chi_3^2 = f^2, \qquad f^2 = 1.$$

Since $f > 0$, we derive $f = 1$, $\chi_1 = \chi_2 = 1$, $\chi_3 = \pm 1$,

If $\chi_3 = 0$, the first four relations give

$$20 = 2f^2 + 4\chi_1^2 + 4\chi_2^2 = 6f^2 + 2f\chi_1 + 2f\chi_2 = 6f^2 - f^2 = 5f^2,$$

whence $f = 2$. The seven relations then reduce to the following:

$$\chi_1^2 = \chi_2 + 2, \ \chi_2^2 = \chi_1 + 2, \ \chi_1 + \chi_2 + 1 = 0, \ \chi_1\chi = \chi_1 + \chi_2,$$

which state merely that χ_1 and χ_2 are the roots of $y^2 + y - 1 = 0$,

$$y_1 = \tfrac{1}{2}(-1 + \sqrt{5}), \quad y_2 = \tfrac{1}{2}(-1 - \sqrt{5}).$$

Hence the four characters are given by the table:

(53)

f	$\chi(E)$	$\chi_1 = \chi(a)$	$\chi_2 = \chi(a^2)$	$\chi_3 = \chi(\beta)$
1	1	1	1	1
1	1	1	1	-1
2	2	y_1	y_2	0
2	2	y_2	y_1	0

Setting $x_0 = x_E$, $x_i = x_{a^i}$ $(i = 1, 2, 3, 4)$, $x'_i = x_{\beta a^i}$ $(i = 0, 1, 2, 3, 4)$, the group-matrix for the dihedron group H_{10} may be written in the abbreviated form

$$\Theta \equiv \left(\begin{array}{c|c} X & X' \\ \hline X' & X \end{array}\right),$$

where

$$X \equiv \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_4 & x_0 & x_1 & x_2 & x_3 \\ x_3 & x_4 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_4 & x_0 \end{pmatrix}, \quad X' \equiv \begin{pmatrix} x'_0 & x'_1 & x'_2 & x'_3 & x'_4 \\ x'_1 & x'_2 & x'_3 & x'_4 & x_0 \\ x'_2 & x'_3 & x'_4 & x'_0 & x'_1 \\ x'_3 & x'_4 & x'_0 & x'_1 & x'_2 \\ x'_4 & x'_0 & x'_1 & x'_2 & x'_3 \end{pmatrix}$$

In view of (53), there are exactly two linear factors of Θ, viz.,

$$(54) \qquad (x_0 + x_1 + x_2 + x_3 + x_4) \pm (x'_0 + x'_1 + x'_2 + x'_3 + x'_4).$$

The two irreducible factors of the second degree, Φ_2 and Φ'_2, may be calculated by means of the results of §7. Let Φ_2 correspond to the character for which $\chi(a) = y_1$, $\chi(a^2) = y_2$. Then Φ'_2 is derived from Φ_2 by interchanging y_1 and y_2. By (19) and (18),

$$2\,\Phi_2 = S_1^2 - S_2,$$

$$S_1 = \sum_R \chi(R)\, x_R \equiv 2\,x_0 + y_1 x_1 + y_1 x_4 + y_2 x_2 + y_2 x_3,$$

$$\begin{aligned} S_2 = \sum_{R_1, R_2} \chi(R_1 R_2)\, x_{R_1} x_{R_2} \\ \equiv 2(x_0^2 + 2\,x_4 x_1 + 2\,x_3 x_2 + x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2) \\ + y_1\,(2\,x_0 x_1 + 2\,x_4 x_2 + x_3^2 + x'_0 x'_1 + x'_1 x'_2 + x'_2 x'_3 + x'_3 x'_4 + x'_4 x'_0) \\ + y_1\,(2\,x_0 x_4 + 2\,x_3 x_1 + x_2^2 + x'_0 x'_4 + x'_1 x'_0 + x'_2 x'_1 + x'_3 x'_2 + x'_4 x'_3) \\ + y_2\,(2\,x_0 x_2 + 2\,x_4 x_3 + x_1^2 + x'_0 x'_2 + x'_1 x_3 + x'_2 x'_4 + x'_3 x'_0 + x'_4 x'_1) \\ + y_2\,(2\,x_0 x_3 + 2\,x_1 x_2 + x_4^2 + x'_0 x'_3 + x'_1 x'_4 + x'_2 x'_0 + x'_3 x'_1 + x'_4 x'_2). \end{aligned}$$

The five expressions in parentheses correspond to the terms of the sum for which $R_1 R_2 = E$, a, a^4, a^2, a^3, respectively. The final result is

$$\begin{aligned} \Phi_2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2 - x_4'^2 + y_1(x_0 x_1 + x_0 x_4 \\ + x_1 x_2 + x_2 x_3 + x_3 x_4) \\ - y_1(x'_0 x'_1 + x'_1 x'_2 + x'_2 x'_3 + x'_3 x'_4 + x'_4 x_0) - y_2(x'_0 x'_2 + x'_1 x'_3 + x'_2 x'_4 + x'_3 x'_0 \\ + x'_4 x'_1) + y_2(x_0 x_2 + x_0 x_3 + x_1 x_3 + x_1 x_4 + x_2 x_4). \end{aligned}$$

As a first check, we note that the coefficient of $x_E'^{-1} x_R \equiv x_0 x_R$ is $\chi(R)$, as required by §7. As a second check, we verify the results of §9. For the special group-matrix, $x_1 = x_4$, $x_2 = x_3$, $x_0' = x_1' = x_2' = x_3' = x_4'$, so that

$$\begin{aligned}\Phi_2 = x_0^2 + 2x_1^2 + 2x_2^2 - 5x_0'^2 + y_1(2x_0x_1 + 2x_1x_2 + x_2^2) - 5y_1x_0'^2 - 5y_2x_0'^2 \\ + y_2(2x_0x_2 + 2x_1x_2 + x_1^2) \equiv (x_0 + y_1x_1 + y_2x_2)^2 = (\tfrac{1}{2}\Sigma\chi(R)x_R)^2 \equiv v_1^2,\end{aligned}$$

upon applying the relations $y_1 y_2 = -1$, $y_1 + y_2 = -1$, and formula (35) for $f = 2$. Hence (34) is verified for the case $u = 0$; and, for the general case, by replacing y_E by $y_E + u$.

17. A table of characters for each of the groups: Tetrahedron H_{12}, Octahedron H_{24}, Icosahedron H_{60}, symmetric group H_{120} on 5 letters, unary modular group of order $\frac{1}{2}p(p^2 - 1)$, are given by Frobenius, Über Gruppencharaktere, *l. c.*, pp. 1011–1021. Frobenius has also determined the characters of the symmetric and alternating groups on n letters, giving explicit tables for the case $n = 8$ (*Berliner Sitzungsberichte*, 1900, pp. 516–534; 1901, pp. 303–315). For any symmetric group, all the values of every character are integral rational numbers.

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